4.3 The Killing form and Cartan's criterion for solvability.

In this short section we briefly introduce the Killing form and discuss its role in the study of solvable Lie algebras via Cartan's criterion.

From now on $\mathfrak{g}$ will be a $\mathbb{K}$ Lie algebra with $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$.

Definition 4.51 [Killing form]
The Killing form of a $\mathbb{K}$-Lie algebra is the bilinear form,

$$K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}.$$

defined by

$$K_{\mathfrak{g}}(X, Y) := \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)).$$

The following invariance property is key for the applications:

Proposition 4.52.

$$K_{\mathfrak g}(\operatorname{ad}(Z)X, Y) + K_{\mathfrak g}(X, \operatorname{ad}(Z)Y) = 0$$
$\forall\, X, Y, Z \in \mathfrak g$

Proof

Recall $\operatorname{ad}(Z)X = [Z, X]$, $\operatorname{ad}(Z)Y = [Z, Y]$.

$K_{\mathfrak g}(\operatorname{ad}(Z)X, Y) + K_{\mathfrak g}(X, \operatorname{ad}(Z)Y) =$

$= \operatorname{tr}(\operatorname{ad}([Z,X])\operatorname{ad}(Y)) + \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}([Z,Y]))$

$= \operatorname{tr}([\operatorname{ad}(Z), \operatorname{ad}(X)]\operatorname{ad}(Y)) + \operatorname{tr}(\operatorname{ad}(X)[\operatorname{ad}(Z), \operatorname{ad}(Y)])$

$= \operatorname{tr}(\operatorname{ad}(Z)\operatorname{ad}(X)\operatorname{ad}(Y)) - \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Z)\operatorname{ad}(Y))$
$\quad + \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Z)\operatorname{ad}(Y)) - \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)\operatorname{ad}(Z))$

$= \operatorname{tr}(\operatorname{ad}(Z)\operatorname{ad}(X)\operatorname{ad}(Y)) - \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)\operatorname{ad}(Z))$

$\rightsquigarrow \operatorname{tr}(AB) = \operatorname{tr}(BA)$.

$= 0$. $\square$

Exercise 4.53

Let G be a connected Lie group with Lie algebra $\mathfrak g$. Prove that

$$K_{\mathfrak g}(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y) = K_{\mathfrak g}(X, Y)$$

for all $g \in G$ and $X, Y \in \mathfrak g$.

Hint: compute the derivative with respect to t of

$K_{\mathfrak{g}}(\mathrm{Ad}(\exp tZ)X, \mathrm{Ad}(\exp tZ)Y)$.

Theorem 4.54 [Cartan's criterion]

A $\mathbb{K}$-Lie algebra is solvable if and only if $K_{\mathfrak{g}}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} = 0$.

We discuss only one implication. The following lemma will be important for us:

Lemma 4.55

Let $\mathfrak{h} \triangleleft \mathfrak{g}$ be an ideal. Then $K_{\mathfrak{g}}|_{\mathfrak{h}\times\mathfrak{h}} = K_{\mathfrak{h}}$.

Proof

Let V be a linear complement of $\mathfrak{h}$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{h} \oplus V$.

If we consider $\mathrm{ad}_{\mathfrak{g}}(X) : \mathfrak{h}\oplus V \to \mathfrak{h}\oplus V$ then:

$\mathrm{ad}_{\mathfrak{g}}(X)Y = [X,Y] \in \mathfrak{h}$ if $Y \in \mathfrak{h}$.

$\mathrm{ad}_{\mathfrak{g}}(X)Y = [X,Y] \in \mathfrak{h}$ if $Y \in V$

since $\mathfrak{h}$ is an ideal. Note that being a subalgebra would be sufficient for the first conclusion.

Hence $ad_{\mathfrak{g}}(x)$ can be represented as

$$ad_{\mathfrak{g}}(x) = \begin{array}{cc} \begin{pmatrix} ad_{\mathfrak{h}}(x) & * \\ 0 & 0 \end{pmatrix} & \begin{array}{c} \mathfrak{h} \\ V \end{array} \\ \begin{array}{cc} \mathfrak{h} & \quad V \end{array} & \end{array}$$

Therefore,

$$\begin{aligned} K_{\mathfrak{g}}(x,y) &= tr(ad_{\mathfrak{g}}(x) ad_{\mathfrak{g}}(y)) \\ &= tr(ad_{\mathfrak{h}}(x)\, ad_{\mathfrak{h}}(y)) \\ &= K_{\mathfrak{h}}(x,y) \end{aligned}$$

for all $x, y \in \mathfrak{h}$. $\square$

Proof of ($\Longrightarrow$) in Theorem 4.54

Assume that $\mathfrak{g}$ is solvable. Then by Theorem 4.38 $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$ is nilpotent. By Corollary 4.44 $ad(\mathfrak{g}^{(1)})$ is strictly upper triangular with respect to some basis. Taking into account Lemma 4.55 above $K_{\mathfrak{g}}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} = K_{\mathfrak{g}^{(1)}} = 0$ since $\mathfrak{g}^{(1)} \triangleleft \mathfrak{g}$. $\square$